MATH 1A - PROOF OF THE FUNDAMENTAL THEOREM OF CALCULUS

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1. The Fundamental Theorem of Calculus

Theorem 1 (Fundamental Theorem of Calculus - Part I)**.** *If f is continuous on $[a,b]$, then the function g defined by:*

$$g(x) = \int_a^x f(t)dt \qquad\qquad a \leq x \leq b$$

is continuous on $[a,b]$, differentiable on (a,b) and $g'(x) = f(x)$

Theorem 2 (Fundamental Theorem of Calculus - Part II)**.** *If f is continuous on $[a,b]$, then:*

$$\int_a^b f(t)dt = F(b) - F(a)$$

where F is any antiderivative of f

2. Proof of FTC - Part I

Let $x \in [a,b]$, let $\epsilon > 0$ and let h be such that $x + h < b$ **AND** $0 < h < \delta$.

Then:

$$\frac{g(x+h) - g(x)}{h} = \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \frac{\int_x^{x+h} f(t)dt}{h}$$

Now, because f is continuous at x, there exists $\delta > 0$ such that, when $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$.

In particular, if $t \in [x, x+h]$, we have $x \leq t \leq x + h$, so $0 < t - x \leq h < \delta$, and so in particular $|t - x| < \delta$, and so we get $|f(t) - f(x)| < \epsilon$.

This implies that $-\epsilon < f(t) - f(x) < \epsilon$, so $f(x) - \epsilon < f(t) < f(x) + \epsilon$.

Integrating this over $[x, x+h]$, and using our comparison inequalities, we get:

$$\begin{aligned} f(x) - \epsilon < f(t) < f(x) + \epsilon \\ \int_x^{x+h} f(x) - \epsilon dt < \int_x^{x+h} f(t)dt < \int_x^{x+h} f(x) + \epsilon dt \\ (f(x) - \epsilon)\int_x^{x+h} dt < \int_x^{x+h} f(t)dt < (f(x) + \epsilon)\int_x^{x+h} dt \end{aligned}$$

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This is because $f(x) - \epsilon$ and $f(x) + \epsilon$ are **constants** with respect to t

$$(f(x) - \epsilon)(x + h - x) < \int_x^{x+h} f(t)dt < (f(x) + \epsilon)(x + h - x)$$

$$(f(x) - \epsilon)h < \int_x^{x+h} f(t)dt < (f(x) + \epsilon)h$$

$$(f(x) - \epsilon) < \frac{\int_x^{x+h} f(t)dt}{h} < (f(x) + \epsilon)$$

$$(f(x) - \epsilon) < \frac{g(x+h) - g(x)}{h} < (f(x) + \epsilon) \qquad \text{(by what we've shown above)}$$

$$-\epsilon < \frac{g(x+h) - g(x)}{h} - f(x) < \epsilon$$

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| < \epsilon$$

And so we've shown that:

$$\lim_{h \to 0^+} \frac{g(x+h) - g(x)}{h} = f(x)$$

Similarly, one can show that:

$$\lim_{h \to 0^-} \frac{g(x+h) - g(x)}{h} = f(x)$$

And hence, we get:

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

But, by definition of a derivative, we have:

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

And so, we finally have:

$$g'(x) = f(x)$$

And we're done! :D

3. Proof of FTC - Part II

This is much easier than Part I!

Let F be an antiderivative of f, as in the statement of the theorem.

Now define a new function g as follows:

$$g(x) = \int_a^x f(t)dt$$

By FTC Part I, g **is continuous on** $[a, b]$ **and differentiable on** (a, b) and $g'(x) = f(x)$ for every x in (a, b).

Now define **another** new function H as follows:

$$h(x) = g(x) - F(x)$$

Then h **is continuous on** $[a, b]$ **and differentiable on** (a, b) as a difference of two functions with those two properties. Moreover, if $x \in (a, b)$, $h'(x) = g'(x) - F'(x)$, but $g'(x) = f(x)$ by FTC Part I, and $F'(x) = f(x)$ by definition of antiderivative. And so $h'(x) = f(x) - f(x) = 0$ **for every** $x \in (a, b)$, and so, because in addition h is continuous at a and b, h is constant on $[a, b]$, and hence $h(a) = h(b)$.

And so, in particular:

$$\begin{aligned}
h(b) &= h(a) \\
g(b) - F(b) &= g(a) - F(a) \qquad \text{(By definition of } h\text{)} \\
g(b) &= g(a) + (F(b) - F(a)) \\
\int_a^b f(t)dt &= \int_a^a f(t)dt + (F(b) - F(a)) \qquad \text{(By definition of } g\text{)} \\
\int_a^b f(t)dt &= 0 + F(b) - F(a) \\
\int_a^b f(t)dt &= F(b) - F(a)
\end{aligned}$$